MATH1520 University Mathematics for Applications **Autumn 2019**

Supplementary 1: Exponential and Logarithmic Functions

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1.1 Family of Exponential Functions

A function of the form $f(x) = b^x$, where $b > 0$, is called an **exponential function with base** b. Some examples are

$$f(x) = 2^x, \qquad f(x) = \left(\frac{1}{2}\right)^x, \qquad f(x) = \pi^x$$

Note that an exponential function has a constant base and variable exponent.

Definition 1.1.1 (Exponential Function)**.** The equation

$$f(x) = b^x \qquad b > 0, b \neq 1$$

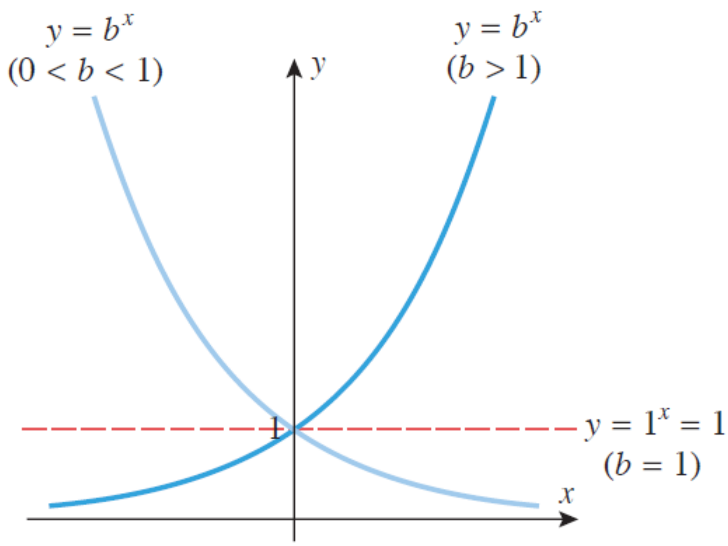
defines an **exponential function** for each different constant b, called the **base**. The **domain** of f is the set of all real numbers, and the **range** of f is the set of all positive numbers.

The following figure illustrates that the graph of $y = b^x$ has one of three general forms, depending on the value of b.

The graph of $y = b^x$ has the following properties:

1. The graph passes through $(0, 1)$ because $b^0 = 1$.
2. If $b > 1$, the value of b^x increases as x increases. The x-axis is a horizontal asymptote of the graph of b^x.
3. If $0 < b < 1$, the value of b^x decreases as x increases. The x-axis is a horizontal asymptote of the graph of b^x .

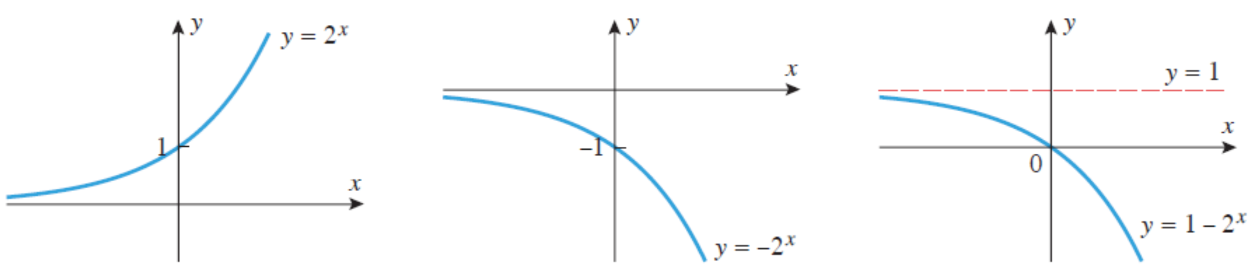
The domain and range of the exponential function $f(x) = b^x$ can also be found by examining the figure.



1. If $b > 0$, then $f(x) = b^x$ is defined and has a real value for every real value of x, so the natural domain of every exponential function is $(-\infty, +\infty)$.
2. If $b > 0$ and $b \neq 1$, then as noted earlier the graph of $y = b^x$ increases indefinitely as it is traversed in one direction and decreases toward zero but never reaches zero as it is traversed in the other direction. This implies that the range of $f(x) = b^x$ is $(0, +\infty)$.

Example 1.1.1. Sketch the graph of the function $f(x) = 1 - 2^x$ and find its domain and range.

Solution. Start with a graph of $y = 2^x$. Reflect this graph across the x-axis to obtain the graph of $y = -2^x$, then translate that graph upward by 1 unit to obtain the graph of $y = 1 - 2^x$. The dashed line in the third part of figure is a horizontal asymptote for the graph. You should be able to see from the graph that the domain of f is $(-\infty, +\infty)$ and the range is $(-\infty, 1)$.



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Theorem 1 (Properties of Exponential Functions)**.** *For a and b positive, $a \neq 1$, $b \neq 1$, and x and y real,*

1. *Exponential laws:*

$$a^x a^y = a^{x+y} \quad \frac{a^x}{a^y} = a^{x-y}$$
$$(a^x)^y = a^{xy} \quad (ab)^x = a^x b^x \quad \left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$$

2. $a^x = a^y$ *if and only if* $x = y$

3. *For* $x \neq 0$, $a^x = b^x$ *if and only if* $a = b$.

1.2 The Natural Exponential Function

Among all possible bases for exponential functions there is one particular base that plays a special role in calculus. That base, denoted by the letter e, is a certain irrational number whose value to six decimal places is

$$e \approx 2.718282$$

The function $f(x) = e^x$ is called the **natural exponential function**.

The constant e also arises in the context of the graph of the equation

$$y = \left(1 + \frac{1}{x}\right)^x$$

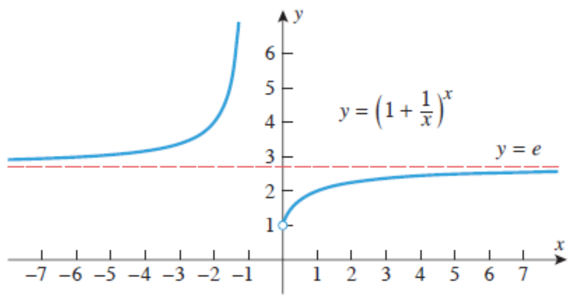
As shown in the following figure $y = e$ is a horizontal asymptote of this graph, i.e.,

$$\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x = e \qquad \lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e$$

1.3 Logarithmic Functions

1.3.1 Inverse Function

Definition 1.3.1 (One-to-One Function)**.** A function f is said to be **one-to-one** if each range value corresponds to exactly one domain value.



APPROXIMATIONS OF e BY $(1+1/x)^x$ FOR INCREASING VALUES OF x

x	$1+\frac{1}{x}$	$\left(1+\frac{1}{x}\right)^x$
1	2	≈ 2.000000
10	1.1	2.593742
100	1.01	2.704814
1000	1.001	2.716924
10,000	1.0001	2.718146
100,000	1.00001	2.718268
1,000,000	1.000001	2.718280

For example, $y = 2x$ is a one-to-one function, but $y = x^2$ is not a one-to-one function.

Definition 1.3.2 (Inverse of a Function)**.** If f is a one-to-one function, then the inverse of f is the function formed by interchanging the independent and dependent variables for f. Thus, if (a, b) is a point on the graph of f, then (b, a) is a point on the graph of the inverse of f.
Note: If f is not one-to-one, then f does not have an inverse.

For example, $y = f(x) = 2x + 1$, then interchanging independent and dependent variables, we have $x = 2y + 1$. Solve the equation for y, we have $y = \frac{x-1}{2}$, which is the inverse function of $f(x)$.

1.3.2 Logarithmic Functions

If we start with the exponential function f defined by

$$y = 2^x$$

and interchange the variables, we obtain the inverse of f:

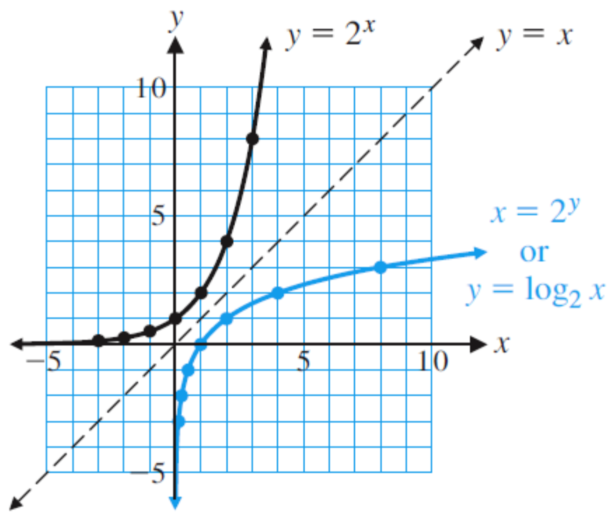
$$x = 2^y$$

We call the inverse of the **logarithmic function with base 2**, and write

$$y = \log_2 x \quad \text{if and only if} \quad x = 2^y$$

The graphs of $y = 2^x$ and $y = \log_2 x$ are shown in the following figure. Note that, if we fold the paper along the dashed ling $y = x$, the two graphs match exactly. The line $y = x$ is a line of symmetry for the two graphs.

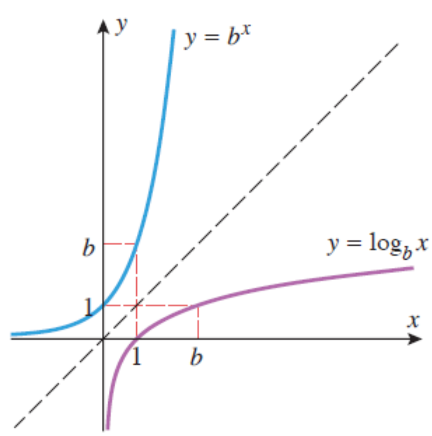
In general, since the graphs of all exponential functions of the form $f(x) = b^x, b \neq 1, b > 0$, are either increasing or decreasing, exponential functions have inverses.



Definition 1.3.3. The inverse of an exponential function is called a **logarithmic function**. For $b > 0$ and $b \neq 1$

$$y = \log_b x \qquad \text{is equivalent to} \qquad x = b^y$$

The log to the base b of x is the exponent to which b must be raised to obtain x.



The **domain** of the logarithmic function is the set of all positive real numbers, which is also the range of the corresponding exponential function; and the **range** of the logarithmic function is the set of all real numbers, which is also the domain of the corresponding exponential function. Typical graphs of an exponential function and its inverse, a logarithmic function, are shown in the following figure.

Example 1.3.1.

$$\begin{array}{ccccc} \log_{10} 100 = 2, & \log_{10}(1/1000) = -3, & \log_2 16 = 4, & \log_b 1 = 0, & \log_b b = 1 \\ 10^2 = 100, & 10^{-3} = 1/1000, & 2^4 = 16, & b^0 = 1, & b^1 = b \end{array}$$

CORRESPONDENCE BETWEEN PROPERTIES OF LOGARITHMIC AND EXPONENTIAL FUNCTIONS	
PROPERTY OF b^x	PROPERTY OF $\log_b x$
$b^0 = 1$	$\log_b 1 = 0$
$b^1 = b$	$\log_b b = 1$
Range is $(0, +\infty)$	Domain is $(0, +\infty)$
Domain is $(-\infty, +\infty)$	Range is $(-\infty, +\infty)$
x-axis is a horizontal asymptote	y-axis is a vertical asymptote

Example 1.3.2. Find y, b, or x, as indicated.

1. Find y: $y = \log_4 16$
2. Find x: $\log_2 x = -3$
3. Find b: $\log_b 100 = 2$

Solution.

1. $y = \log_4 16$ is equivalent to $16 = 4^y$. So,

$$y = 2$$

2. $\log_2 x = -3$ is equivalent to $x = 2^{-3}$. So,

$$x = \frac{1}{2^3} = \frac{1}{8}$$

3. $\log_b 100 = 2$ is equivalent to $100 = b^2$. So,

$$b = 10$$

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Theorem 2 (Properties of Logarithmic Functions)**.** *If b, M, and N are positive real numbers, and p and x are real numbers, then*

1. $\log_b 1 = 0$

2. $\log_b b = 1$

3. $\log_b b^x = x$

4. $b^{\log_b x} = x, \quad x > 0$
5. $\log_b MN = \log_b M + \log_b N$
6. $\log_b \frac{M}{N} = \log_b M - \log_b N$
7. $\log_b M^p = p \log_b M$
8. $\log_b M = \log_b N$ *if and only if* $M = N$

Remark.

Common logarithm: $\log x$ means $\log_{10} x$

Natural logarithm: $\ln x$ means $\log_e x$

So,

$$\begin{aligned} \log x = y \quad &\text{is equivalent to} \quad x = 10^y \\ \ln x = y \quad &\text{is equivalent to} \quad x = e^y \end{aligned}$$

Example 1.3.3.

1. $\log_b \dfrac{wx}{yz} = \log_b w + \log_b x - \log_b y - \log_b z$
2. $\log_b (wx)^{3/5} = \dfrac{3}{5} \log_b (wx) = \dfrac{3}{5} (\log_b w + \log_b x)$
3. $e^{x \ln b} = e^{\ln b^x} = b^x$
4. $\dfrac{\ln x}{\ln b} = \dfrac{\ln(b^{\log_b x})}{\ln b} = \dfrac{(\log_b x)(\ln b)}{\ln b} = \log_b x$ (change of base formula)

Theorem 3 (Useful property of natural logarithms)**.**

$$e^{x \ln b} = b^x \qquad \textit{and} \qquad \frac{\ln x}{\ln b} = \log_b x$$